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Exceptional sets of hypergeometric series

Natália Archinard

McGill University, 805 Sherbrooke St. W., Montreal, QC, Canada H3A-2K6

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Abstract

For a family of transcendental hypergeometric series, we determine explicitly the set of algebraic points at which the series takes algebraic values (the so-called *exceptional set*). This answers a question of Siegel in special cases. For this, we first prove identities, each one relating locally one hypergeometric series to modular functions. In some cases, the identity and the theory of complex multiplication allow the determination of an infinite subset of the exceptional set. These subsets are shown to be the whole sets in using a consequence of Wüstholz's Analytic Subgroup Theorem together with mapping properties of Schwarz triangle functions. Further consequences of the identities are explicit evaluations of hypergeometric series at algebraic points. Some of them provide examples for Kroneckers Jugendtraum.

MSC: 11

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Introduction

The Gauss' hypergeometric series

$$F(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a;n)(b;n)}{(c;n)(1;n)}z^n,$$

where $(x;n) := \prod_{j=1}^{n}(x+n-j)$ and $a,b,c\in\mathbb{Q}$, $-c\notin\mathbb{N}$, defines an analytic function of z on the unit disc and satisfies a linear differential equation with monodromy

E-mail address: archinard@math.mcgill.ca.

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group $\Delta(a,b,c)$. Schwarz [20] showed that this function is algebraic exactly when its monodromy group is finite. Hence, for most values of the parameters a,b,c, this series defines a transcendental function. As for other G-functions, it has been believed that transcendental hypergeometric series would take transcendental values at algebraic points. Siegel [22] asked the question of the cardinality and of the nature of the so-called *exceptional set*:

$$E(a,b,c) := \{x \in \bar{\mathbb{Q}};\ F(a,b,c;z) \in \bar{\mathbb{Q}}\}.$$

Using a consequence ([29, Satz 2]) of Wüstholz's Analytic Subgroup Theorem [30], Wolfart [27,28] showed, with some conditions on a,b,c, that the points z in $E(a,b,c)$ correspond to isogenous abelian varieties $T(z)$ of the same dimension, which are defined over $\bar{\mathbb{Q}}$ and have complex multiplication (CM) in the sense of Shimura–Taniyama (see further details in [10]). Let $Z(a,b,c)$ be the closure of the image under $z \mapsto T(z)$ of $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ on the Siegel moduli variety of such abelian varieties, as studied for example by Cohen and Wolfart [8]. The study of $E(a,b,c)$ is then reduced to that of the CM points on $Z(a,b,c)$, that is, of the points corresponding to abelian varieties with CM. Along these lines, Wolfart proved the infiniteness of the exceptional set $E(a,b,c)$ in the cases where the monodromy group $\Delta(a,b,c)$ is arithmetic and the conditions $c<1$, $c>a>0$, $c>b>0$, $|1-c|$, $|c-a-b|$, $|a-b|$ $\in \frac{1}{\mathbb{Z}} \cup \{0\}$ and $|1-c| + |c-a-b| + |a-b| < 1$ hold. Under the latter conditions, the monodromy group is triangular. The 85 conjugacy classes of arithmetic triangle groups were determined by Takeuchi [24]. Wolfart also asserted that the exceptional set can only be infinite when the monodromy group is arithmetic, but a gap in his proof was found by Gubler. Cohen and Wüstholz [10] proposed a weak version of André–Oort conjecture to show that a nonarithmetic group implies a finite exceptional set. This weak conjecture for an irreducible variety Z of dimension 1 states roughly that if Z contains a Zariski dense subset of CM points corresponding to isogenous abelian varieties, then Z is of Hodge type (for a precise statement, see [10]). It was proved recently by Edixhoven and Yafaev [12]. An exceptional set for Appell–Lauricella hypergeometric series in two variables is defined by Desrousseaux [11] in an analogous way, so that the points of the exceptional set correspond to abelian varieties with CM. Part of Desrousseaux's work relies on a weak André–Oort conjecture for CM points on surfaces in Shimura varieties as given in Cohen [7]. After the result of Wolfart et al. [6], Flach [13], Joyce and Zucker [16] published on special algebraic values of hypergeometric series with monodromy group isomorphic to $SL_2(\mathbb{Z})$. Properties of v-adic evaluations of hypergeometric series were also studied by Beukers [5] and André [2].

The purpose of the present article is the explicit determination of the exceptional sets (Theorem 5.4) for a family of hypergeometric series whose monodromy group is isomorphic to $SL_2(\mathbb{Z})$, namely the family of hypergeometric series which appear in Kummer's solutions to the Picard–Fuchs differential equation of the universal family of elliptic curves. These series are transcendental functions and there are 16 of them. The exceptional sets will be shown to be trivial in the cases where $c>1$ (Section 5). In the cases with $c<1$, the exceptional set is infinite by Wolfart's result. In these cases, a necessary condition on z to lie in $E(a,b,c)$ will be worked out (Section 5) by applying a result of Wüstholz on linear independence of periods to Euler's integral representation of the hypergeometric series viewed as a quotient of periods on abelian varieties defined over $\bar{\mathbb{Q}}$ (this idea is due to Wolfart [28]). The construction of the abelian varieties will be made precise in Section 2. The exact knowledge of the lattice of these complex tori up to isogeny (Section 2) together with that of the image of the Schwarz maps (Section 1) are essential for the translation of the necessary condition in usable terms (Section 5). On the other hand, the sufficience of these necessary conditions on z will be shown in Section 4.1 via *identities*. These identities (Section 3) relate locally one hypergeometric series to the modular functions J and η^2. They contain information on the arithmetic nature of the hypergeometric series. For instance, they allow us to calculate explicit evaluations of hypergeometric series at algebraic points, some of them providing examples for Kroneckers Jugendtraum (Sections 4.2 and 4.3). For this, we first have to compute the values of $J(mi)$ and $(\frac{\eta(mi)}{\eta(i)})^2$ for some m (Section 4.2). Some results of Sections 2, 3, 4.1 and 4.2 are taken from the author's Ph.D. thesis [4].

1. Hypergeometric series and Schwarz triangle functions

The Gauss' hypergeometric series $F(a,b,c;z)$ with $a,b,c\in\mathbb{C}$, $-c\notin\mathbb{N}$ has an integral representation due to Euler, which reads, for $|z|<1$,

$$F(a,b,c;z) = \frac{1}{B(b,c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx, \tag{1}$$

where $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ is the Beta-function. The two integrals converge for $Re(c)>Re(b)>0$. $F(a,b,c;z)$ is the holomorphic solution taking value 1 at 0 to the so-called *hypergeometric differential equation*

$$Equ(a,b,c): z(1-z)\frac{d^2u}{dz^2}+(c-(a+b+1)z)\frac{du}{dz}-abu=0. \tag{2}$$

The only singular points of $Equ(a,b,c)$ lie at $0,1,\infty$ and are regular singularities. The monodromy group of $Equ(a,b,c)$ will be denoted by $\Delta(a,b,c)$. If $a,b,c\in\mathbb{R}$, $|1-c|,|c-a-b|,|a-b|\in\frac{1}{\mathbb{Z}}\cup\{0\}$ and $|1-c|+|c-a-b|+|a-b|<1$ hold, then $\Delta(a,b,c)$ is a triangle subgroup of $SL_2(\mathbb{R})$ of type given up to permutation by $(\frac{1}{|1-c|},\frac{1}{|c-a-b|},\frac{1}{|a-b|})$. If $a,b,c\in\mathbb{R}$ and $|1-c|,|c-a-b|,|a-b|<1$, it is classically known (see [19, V.7, VI.5]) that the quotient of two linearly independent solutions to the hypergeometric differential equation maps a complex half plane bijectively to the interior of a triangle whose sides are arcs of circles. Such a map is called a *Schwarz triangle map* and the triangle a *Schwarz triangle*. The real line is mapped to the boundary of the Schwarz triangle and the points $0,1,\infty$ to the vertices. In particular, if $a,b,c\in\mathbb{Q}$, $|1-c|,|c-a-b|,|a-b|\in\frac{1}{\mathbb{Z}}\cup\{0\}$ and $|1-c|+|c-a-b|+|a-b|<1$, the Schwarz triangle map

$$D_{abc}(z) := (-1)^{1+a+b-c} z^{1-c} \frac{B(1+a-c, 1-a)F(1+a-c, 1+b-c, 2-c; z)}{B(b, c-b)F(a, b, c; z)}$$

maps the upper half plane to a Schwarz triangle with angles

$$\pi|1-c| \quad \text{at } D_{abc}(0),$$

$$\pi|c-a-b| \quad \text{at } D_{abc}(1),$$

$$\pi|a-b| \quad \text{at } D_{abc}(\infty).$$

If $c \neq 1$, we have trivially $D_{abc}(0) = 0$. In order to calculate the vertices at the images of 1 and ∞, we need to know the analytic continuation of $F(a, b, c; z)$ at these points. The analytic continuation at 1 is given by (cf. [26, 14.11])

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \text{if } c-a-b > 0$$

and for $z \to 1-0$ by (cf. [26, XIV ex.18])

$$F(a, b, c; z) \sim \begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log(\frac{1}{1-z}) & \text{if } c-a-b = 0, \\ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} & \text{if } c-a-b < 0. \end{cases}$$

Concerning the analytic continuation towards ∞, we have, if $|z| > 1$, $|arg(-z)| < \pi$, $a, b \notin \mathbb{Z}_{\leqslant 0}$ and $a-b \notin \mathbb{Z}$ (cf. [1, 15.3.7])

$$\begin{aligned} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b, c; z) = & \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)} (-z)^{-a} F(a, 1-c+a, 1-b+a; z^{-1}) \\ & + \frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)} (-z)^{-b} F(b, 1-c+b, 1-a+b; z^{-1}) \end{aligned} \tag{3}$$

and, if moreover, $c-a \notin \mathbb{Z}$ (cf. [1, 15.3.13])

$$\begin{aligned} F(a, a, c; z) = & \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} (-z)^{-a} \sum_{n=0}^{\infty} \frac{(a;n)(1-c+a;n)}{(n!)^2} z^{-n}. \\ & \cdot [\ln(-z) + 2\psi(n+1) - \psi(a+n) - \psi(c-a-n)], \end{aligned} \tag{4}$$

where $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the logarithmic derivative of $\Gamma(z)$. Using this and some classical properties of the Gamma function (e.g. [26, XII]), we calculate

$$D_{abc}(1)=\begin{cases}(-1)^{1-c+a+b}\,\frac{\sin(b\pi)}{\sin((c-a)\pi)} & \text{if } c-a-b>0,\\ -1 & \text{if } c-a-b=0,\\ (-1)^{1-c+a+b}\,\frac{\sin((c-b)\pi)}{\sin(a\pi)} & \text{if } c-a-b<0\end{cases}$$

and, if $a,b\notin\mathbb{Z}_{\leqslant 0}$,

$$D_{abc}(\infty)=\begin{cases}1 & \text{if } a=b,\ c-a\notin\mathbb{Z},\\ (-1)^{a+b}\,\frac{\sin((c-b)\pi)}{\sin((c-a)\pi)} & \text{if } b-a>0,\ a-b\notin\mathbb{Z},\\ (-1)^{a+b}\,\frac{\sin(b\pi)}{\sin(a\pi)} & \text{if } b-a<0,\ a-b\notin\mathbb{Z}.\end{cases}$$

These results will be used in Section 5 in order to determine explicitly the Schwarz triangles for the considered tiples (a,b,c), see table (23).

2. Construction of Abelian varieties

Following the idea of Wolfart [27,28], we write Euler's integral representation (1) of the hypergeometric series as

$$F(a,b,c;z)=\frac{\mathcal{P}(z)}{\mathcal{P}(0)},$$

where $Re(c)>Re(b)>0$ and $\mathcal{P}(z)=\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx$. If $b,c-b\notin\mathbb{Z}$, this integral can be replaced up to an algebraic factor by an integral along a double contour loop $\gamma_{0,1}$ (or *Pochhammer loop*, cf. [31, IV.2.2]). $\mathcal{P}(z)$ can then be viewed up to an algebraic factor as the integral of a differential form $\frac{dx}{y}$ on a nonsingular algebraic curve and, if $\frac{dx}{y}$ is regular, as a period on the Jacobian variety of this curve. More precisely, for $z\in\mathbb{P}^1_{\mathbb{C}}-\{0,1,\infty\}$, let $C_{abc}(N,z)$ be the projective algebraic curve defined affinely by the equation

$$y^N=x^A(1-x)^B(1-zx)^C,$$

where $N=lcd(a,b,c)$, $A=N(1-b)$, $B=N(1+b-c)$ and $C=Na$. By symmetry, we assume the following hypotheses:

$$a,b,c\in\mathbb{Q}-\mathbb{Z},\quad c-a,c-b\notin\mathbb{Z},$$

which imply

$$N\nmid A,B,C,A+B+C.$$

Moreover, we will suppose $C_{abc}(N,z)$ to be irreducible, that is $(N,A,B,C)=1$. In general, $C_{abc}(N,z)$ is singular at the points $(x,y)=(0,0)$, $(1,0)$, $(\frac{1}{z},0)$, ∞. We then let $X_{abc}(N,z)$ be its desingularization and $\pi: X_{abc}(N,z)\to C_{abc}(N,z)$ be the desingularization morphism. Both curves are defined over $\mathbb{Q}(z)$. $C_{abc}(N,z)$ carries an action of the group μ_N of Nth roots of unity, which is affinely given by $\zeta\cdot(x,y) := (x,\zeta^{-1}y)$. This action induces an action of μ_N on the vector space of differential 1-forms on $C_{abc}(N,z)$. It is a classical fact that a basis of regular differential 1-forms on $X_{abc}(N,z)$ is given by the regular pull-backs $\pi^*\omega_n$ of the eigenforms

$$\omega_n(x,y) := y^{-n}x^{b_0}(1-x)^{b_1}(1-zx)^{b_2}\,dx$$

on $C_{abc}(N,z)$, where $n\in\{0,\dots,N-1\}$, $a_i\in\mathbb{Z}$.

The explicit construction of $X_{abc}(N,z)$ by local blow-ups and the expressions of π in local coordinates were carried out in [4, 1.4]. With them, the genus was calculated to be [4, 1.5.3]

$$g[X_{abc}(N,z)] = N+1-\tfrac{1}{2}[(N,A)+(N,B)+(N,C)+(N,N-A-B-C)]$$

and the conditions for the regularity of $\pi^*\omega_n$ on $X_{abc}(N,z)$ to read

$$\begin{aligned} b_0 &\geqslant \frac{nA+(N,A)}{N}-1\\ b_1 &\geqslant \frac{nB+(N,B)}{N}-1\\ b_2 &\geqslant \frac{nC+(N,C)}{N}-1\\ b_0+b_1+b_2 &\leqslant \frac{n(A+B+C)-(N,N-A-B-C)}{N}-1. \end{aligned} \tag{5}$$

The vector space $\Gamma(X_{abc}(N,z),\Omega^1)$ of regular differential 1-forms splits into a sum $\bigoplus_{0\leqslant n<N}V_n$ of subrepresentations for the action of μ_N. If $(n,N)=1$, the dimension of the isotypical component V_n of character $\chi_n:\zeta\mapsto\zeta^n$ is equal to

$$d_n := \left\langle\frac{nA}{N}\right\rangle+\left\langle\frac{nB}{N}\right\rangle+\left\langle\frac{nC}{N}\right\rangle-\left\langle\frac{n(A+B+C)}{N}\right\rangle \tag{6}$$

if $d_n>0$ and to 0 otherwise. For $(n,N)=1$, one can show

$$dim\ V_n + dim\ V_{-n} = 2.$$

Remark 1. If $d|N$, let $\psi_d: C_{abc}(N,z)\to C_{abc}(d,z)$ denote the morphism given by $(x,y)\mapsto(x,y^{\frac{N}{d}})$. We have

$$\bigoplus_{\substack{(n,N)=1\\ 0\leqslant n<N}} V_n = \bigcap_{\substack{d|N\\ 0\leqslant d<N}} \pi^*(\ker\Gamma\psi_d).$$

In view of Remark 1, the elements of V_n with $(n,N)=1$ are said to be *new*. The vector subspace $\Gamma(X_{abc}(N,z),\Omega^1)_{\text{new}} \coloneqq \oplus_{(n,N)=1} V_n$ defines an abelian subvariety $T_{abc}(z)$ of the Jacobian variety $\Gamma(X_{abc}(N,z),\Omega^1)^*/\imath(H_1(X_{abc}(N,z),\mathbb{Z}))$, where $\imath$ is the injection given by $[\gamma]\mapsto[\omega\mapsto\int_\gamma\omega]$. The dimension of $T_{abc}(z)$ equals

$$dim(\Gamma(X_{abc}(N,z),\Omega^1)_{\text{new}}) = \frac{1}{2}\sum_{(n,N)=1}(dim\ V_n + dim\ V_{-n}) = \varphi(N).$$

Note that $T_{abc}(z)$ is defined over $\mathbb{Q}(z)$ and in particular over $\bar{\mathbb{Q}}$ when $z\in\bar{\mathbb{Q}}$. Its endomorphism algebra contains $\mathbb{Q}(\zeta_N)$, where $\zeta_N \coloneqq e^{\frac{2\pi i}{N}}$. All these results are proved in [4].

Remark 2. The idea of this construction is due to Wolfart in [28], where the dimension is given (see also [8]). Abelian varieties associated to Appell–Lauricella hypergeometric series in r variables are defined by intersection of kernels (cf. Remark 1) in [9] for $r=2$ and in [21] for any $r\geqslant 1$. The Beta-function corresponds to the case $r=0$. Gross and Rohrlich [14] defined abelian varieties associated to it as suitable quotients of the Jacobian.

Remark 3. The construction of this section was generalized in [4, 1.4] (see also [3]) for Appell–Lauricella hypergeometric series $F(a,b_1,\ldots,b_r;c;\lambda_1,\ldots,\lambda_r)$ in r variables. In this case, the new part of the Jacobian has dimension $\frac{(r+1)\varphi(N)}{2}$ and its endomorphism algebra contains $\mathbb{Q}(\zeta_N)$. N denotes here the least common denominator of $c-\sum_{j=1}^r b_j, 1+a-c, b_1,\ldots,b_r$.

Remark 4. The abelian variety associated to $B(\alpha,\beta)$ has dimension $\frac{\varphi(M)}{2}$ and its algebra of endomorphisms contains $\mathbb{Q}(\zeta_M)$, where $M=lcd(\alpha,\beta)$ (e.g. [4, 1.6]).

By construction, $T_{abc}(z)$ is isomorphic to the quotient of $\mathbb{C}^{\varphi(N)}$ by the lattice of periods. For explicit computations, it is useful to know this lattice explicitly (up to isogeny). The dimension of $T_{abc}(z)$ being $\varphi(N)$, its first homology group has rank $2\varphi(N)$ over $\mathbb{Z}$. We know that the double contour loops γ_{01} and $\gamma_{\frac{1}{z}\infty}$ lift to $X_{abc}(N,z)$, thus they lift also to $T_{abc}(z)$. The lifts of γ_{01} (respectively those of $\gamma_{\frac{1}{z}\infty}$) are permuted under $\mathbb{Z}[\zeta_N]$ and the lifts of γ_{01} are independent over $\mathbb{Z}[\zeta_N]$ to those of $\gamma_{\frac{1}{z}\infty}$. Hence, the lifts of γ_{01} and $\gamma_{\frac{1}{z}\infty}$ together generate a $\mathbb{Z}[\zeta_N]$-module of rank 2 having finite index in $H_1(T_{abc}(z),\mathbb{Z})$. Let σ_n be the automorphism of $\mathbb{Z}[\zeta_N]$ defined by $\zeta_N\mapsto\zeta_N^n$ and set

$$\Lambda_{abc}(z) \coloneqq \left\{ \left(\sigma_n(u)\int_{\gamma_{01}}\omega_n + \sigma_n(v)\int_{\gamma_{\frac{1}{z}\infty}}\omega_n \right)_{\substack{\omega_n \text{ basis el.} \\ \text{of } V_n,\ (n,N)=1}} ; u,v\in\mathbb{Z}[\zeta_N] \right\},$$

then $\mathbb{C}^{\varphi(N)}/\Lambda_{abc}(z)$ is isogenous to $T_{abc}(z)$.

3. Identities

This section is a concentrated version of [4, 2.5]. The Picard–Fuchs differential equation of the universal family of elliptic curves reads [17, II, 3]

$$\frac{d^2\Omega}{dJ^2} + \frac{1}{J}\frac{d\Omega}{dJ} + \frac{31J-4}{144J^2(1-J)^2}\Omega = 0. \tag{7}$$

The solutions to (7) are those to the hypergeometric differential equation $Equ(\frac{11}{12}, \frac{11}{12}, \frac{4}{3})$ multiplied by $J^{\frac{1}{6}}(1-J)^{\frac{3}{4}}$. The monodromy group of (7) has type $(2, 3, \infty)$. By Kummer's method (e.g. [15, Section 2.1.3]), one can compute all solutions to (7) having the form $J^\alpha(1-J)^\beta F(a, b, c; v(J))$, where $v(J)$ is one of the transformations $\{J, \frac{1}{J}, 1-J, \frac{1}{1-J}, \frac{J}{J-1}, \frac{J-1}{J}\}$ preserving the set of singular points $\{0, 1, \infty\}$. We order below the solutions to (7) by groups of solutions equal to $\Omega_{P,\upsilon}$, where P is a singular point and υ the characteristic exponent of $Equ(\frac{11}{12}, \frac{11}{12}, \frac{4}{3})$ at P.

$$\begin{aligned}
\Omega_{0,0}(J) &= \begin{cases} J^{\frac{1}{6}}(1-J)^{\frac{3}{4}} & F(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}; J) \\ J^{\frac{1}{6}}(1-J)^{\frac{1}{4}} & F(\frac{5}{12}, \frac{5}{12}, \frac{4}{3}; J) \\ J^{\frac{1}{6}}(1-J)^{-\frac{1}{6}} & F(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; \frac{J}{J-1}) \end{cases} \\
\Omega_{0,1-c}(J) &= \begin{cases} J^{-\frac{1}{6}}(1-J)^{\frac{3}{4}} & F(\frac{7}{12}, \frac{7}{12}, \frac{2}{3}; J) \\ J^{-\frac{1}{6}}(1-J)^{\frac{1}{4}} & F(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}; J) \\ J^{-\frac{1}{6}}(1-J)^{\frac{1}{6}} & F(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}; \frac{J}{J-1}) \end{cases} \\
\Omega_{1,0}(J) &= \begin{cases} J^{\frac{1}{6}}(1-J)^{\frac{3}{4}} & F(\frac{11}{12}, \frac{11}{12}, \frac{3}{2}; 1-J) \\ J^{-\frac{1}{6}}(1-J)^{\frac{3}{4}} & F(\frac{7}{12}, \frac{7}{12}, \frac{3}{2}; 1-J) \\ J^{-\frac{3}{4}}(1-J)^{\frac{3}{4}} & F(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; \frac{J-1}{J}) \end{cases} \\
\Omega_{1,c-a-b}(J) &= \begin{cases} J^{\frac{1}{6}}(1-J)^{\frac{1}{4}} & F(\frac{5}{12}, \frac{5}{12}, \frac{1}{2}; 1-J) \\ J^{-\frac{1}{6}}(1-J)^{\frac{1}{4}} & F(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}; 1-J) \\ J^{-\frac{1}{4}}(1-J)^{\frac{1}{4}} & F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{J-1}{J}) \end{cases} \\
\Omega_{\infty,a}(J) = \Omega_{\infty,b}(J) &= \begin{cases} J^{-\frac{1}{4}}(1-J)^{\frac{1}{4}} & F(\frac{1}{12}, \frac{5}{12}, 1; \frac{1}{J}) \\ J^{-\frac{3}{4}}(1-J)^{\frac{3}{4}} & F(\frac{7}{12}, \frac{11}{12}, 1; \frac{1}{J}) \\ J^{-\frac{1}{6}}(1-J)^{\frac{1}{6}} & F(\frac{1}{12}, \frac{7}{12}, 1; \frac{1}{1-J}) \\ J^{\frac{1}{6}}(1-J)^{-\frac{1}{6}} & F(\frac{5}{12}, \frac{11}{12}, 1; \frac{1}{1-J}). \end{cases}
\end{aligned} \tag{8}$$

On the other hand, a fundamental system of solutions is given by linearly independent periods on the universal elliptic curve

$$E_J : y^2 = 4x^3 - \frac{27J}{J-1}(x+1),$$

where J is the modular invariant normalized by $J(i) = 1$. Since the parameter space $\mathbb{H}/SL_2(\mathbb{Z})$ of isomorphy classes of elliptic curves over $\mathbb{C}$ is isomorphic to $\mathbb{C}$ under J, for each $\tau \in \mathbb{H}$ such that $J(\tau) = J$, there is an isomorphism $\varphi_\tau : E_\tau \to E_J$ making the following diagram commute:

$$\begin{tikzcd} E_\tau(\mathbb{C}) \arrow[r, "\sim"] & E_J(\mathbb{C}) \\ \mathbb{C}/\Lambda_\tau \arrow[u, "(1:\wp_{\Lambda_\tau}:\wp'_{\Lambda_\tau})", "\wr"'] \arrow[r, "\sim"] & \mathbb{C}/\Lambda_J, \arrow[u, "\wr", "(1:\wp_{\Lambda_J}:\wp'_{\Lambda_J})"'] \end{tikzcd}$$

where $E_\tau : \tilde{y}^2 = 4\tilde{x}^3 - g_2(\tau)\tilde{x} - g_3(\tau)$, g_k being the Eisenstein series of weight $2k$ and $\wp_\Lambda$ the Weierstrass $\wp$-function associated to a lattice Λ. For such a τ, there exists $\mu(\tau) \in \mathbb{C}$ such that $\Lambda_J = \mu(\tau)\Lambda_\tau$. Checking that φ_τ is given by $(x, y) = (\frac{g_2(\tau)}{g_3(\tau)}\tilde{x}, (\frac{g_2(\tau)}{g_3(\tau)})^{\frac{3}{2}}\tilde{y})$, we find

$$\mu(\tau) = \sqrt{\frac{g_3(\tau)}{g_2(\tau)}} = \frac{1}{27}J(\tau)^{-\frac{1}{6}}(1 - J(\tau))^{\frac{1}{4}}\Delta(\tau)^{\frac{1}{12}},$$

where $\Delta = g_2^3 - 27g_3^2$ is the discriminant function. The functions $\mu(\tau)$ and $\tau\mu(\tau)$ are $\mathbb{C}$-linearly independent and multivalued functions of J. Thus, their branches form local systems of solutions to (7). Hence, for each solution to (7) of the form $J^\alpha(1 - J)^\beta F(a, b, c; v(J))$, there exist $A, B \in \mathbb{C}$ such that

$$F(a, b, c; v(J)) = (A\tau + B)J(\tau)^{-\frac{1}{6}-\alpha}(1 - J(\tau))^{\frac{1}{4}-\beta}\Delta(\tau)^{\frac{1}{12}}. \tag{9}$$

Such a relation holds locally at any $\tau_0 \in \mathbb{H}$ satisfying $v(J)(\tau_0) = 0$. For such a τ_0, we define $C_{v(J),\tau}$ to be the connected component of $\{\tau \in \mathbb{H}; |v(J(\tau))| < 1\}$ which contains τ_0. Such a domain is simply connected and invariant under the action by fractional transformation of the elements of $SL_2(\mathbb{Z})$ which fix τ_0. In the following charts, γ_{τ_0} will denote such a modular transformation. On such a domain, the constants and the branches of the roots can therefore be determined. In view of the relation $\Delta = (2\pi)^{12}\eta^{24}$, where η is the Dedekind Eta function, $\Delta^{\frac{1}{12}}$ can be replaced on $C_{v(J),\tau}$ by η^2 up to a complex constant.

3.1. Solutions at $J = 1$

For these solutions, we choose $\tau_0 = i$. In the cases with $c = \frac{3}{2}$, we have $\frac{1}{4} - \beta = -\frac{1}{2}$. The left-hand side of (9) being holomorphic at $\tau_0 = i$, the right-hand side must also be so. As J

takes value 1 at i with multiplicity 2, $(1-J(\tau))^{-\frac{1}{2}}$ has a pole of order 1 at i. Consequently, $A\tau+B$ must have a zero of order 1 at i. This means that $B=-iA$ must hold.

In the cases with $c=\frac{1}{2}$, a relation between the constants can be found by applying the modular transformation $\tau\mapsto -\frac{1}{\tau}$ to the identity. The invariance of the domain and of the function J under this transformation together with the property $\eta(-\frac{1}{\tau})=\sqrt{-i\tau}\eta(\tau)$ imply the relation $B=iA$.

In each case, the value of the constant A is then determined by inserting $\tau=i$ in the identity. The branches of the roots are real positive for real positive arguments. We obtain the following results.

c	solutions	β	τ_0	γ_{τ_0}	B	A^{-1}
$\frac{1}{2}$	$\Omega_{1,c-a-b}(J)$	$\frac{1}{4}$	i	$\tau\mapsto -\frac{1}{\tau}$	iA	$2i\eta(i)^2$
$\frac{3}{2}$	$\Omega_{1,0}(J)$	$\frac{3}{4}$		no need	$-iA$	$res_i((1-J)^{-\frac{1}{2}})\eta(i)^2$

This implies the following identities.

$\boldsymbol{c=\frac{1}{2}}$

For $\tau\in C_{1-J,i}$ and $\tau\in C_{\frac{J-1}{J},i}$ respectively, we have

$$\begin{aligned} F\left(\frac{1}{12},\frac{1}{12},\frac{1}{2};1-J(\tau)\right) &= \frac{\tau+i}{2i}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2 \\ F\left(\frac{5}{12},\frac{5}{12},\frac{1}{2};1-J(\tau)\right) &= \frac{\tau+i}{2i}J(\tau)^{-\frac{1}{3}}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2 \\ F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{J(\tau)-1}{J(\tau)}\right) &= \frac{\tau+i}{2i}J(\tau)^{\frac{1}{12}}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2. \end{aligned} \tag{10}$$

Note 1. The identity for $F(\frac{1}{12},\frac{5}{12},\frac{1}{2};z)$ was known by Beukers and Wolfart [6].

$\boldsymbol{c=\frac{3}{2}}$

For $\tau\in C_{1-J,i}$ and $\tau\in C_{\frac{J-1}{J},i}$ respectively, we have

$$\begin{aligned} F\left(\frac{7}{12},\frac{7}{12},\frac{3}{2};1-J(\tau)\right) &= res_i((1-J)^{-\frac{1}{2}})^{-1}(\tau-i)(1-J(\tau))^{-\frac{1}{2}}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2 \\ F\left(\frac{11}{12},\frac{11}{12},\frac{3}{2};1-J(\tau)\right) &= res_i((1-J)^{-\frac{1}{2}})^{-1}(\tau-i)(1-J(\tau))^{-\frac{1}{2}}J(\tau)^{-\frac{1}{3}}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2 \\ F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2};\frac{J(\tau)-1}{J(\tau)}\right) &= res_i((1-J)^{-\frac{1}{2}})^{-1}(\tau-i)(1-J(\tau))^{-\frac{1}{2}}J(\tau)^{\frac{7}{12}}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2. \end{aligned} \tag{11}$$

3.2. Solutions at $J = 0$

Here, we work out identities in the neighbourhoods of $\rho := e^{\frac{2\pi i}{3}}$ and resp. $-\bar{\rho}$. It goes very similarly as for the solutions at $J = 1$.

The cases with $c = \frac{4}{3}$ correspond to $\alpha = \frac{1}{6}$. Since the function J takes value 0 with multiplicity 3 at any point congruent to ρ modulo $SL_2(\mathbb{Z})$, $J^{-\frac{1}{3}}$ has a pole of order 1 at any such point. It follows the relations $B = -\rho A$ at $\tau_0 = \rho$ and $B = \bar{\rho} A$ at $\tau_0 = -\bar{\rho}$.

In the cases with $c = \frac{2}{3}$, we apply a modular transformation γ_{τ_0}, which fixes τ_0. This transformation together with the obtained relation are given in the following table. The constant A is then determined by setting $\tau = \tau_0$ in the identity. The branches of the roots are real positive for real positive arguments.

c	solutions	α	τ_0	γ_{τ_0}	B	A^{-1}
$\frac{2}{3}$	$\Omega_{0,1-c}(J)$	$-\frac{1}{6}$	ρ	$\tau \mapsto -\frac{1}{\tau+1}$	$-\bar{\rho}A$	$i\sqrt{3}\ \eta(\rho)^2$
			$-\bar{\rho}$	$\tau \mapsto -\frac{\tau-1}{\tau}$	ρA	$i\sqrt{3}\ \eta(-\bar{\rho})^2$
$\frac{4}{3}$	$\Omega_{0,0}(J)$	$\frac{1}{6}$	ρ	no need	$-\rho A$	$res_{\rho}(J^{-\frac{1}{3}})\eta(\rho)^2$
			$-\bar{\rho}$		$\bar{\rho}A$	$res_{-\bar{\rho}}(J^{-\frac{1}{3}})\eta(-\bar{\rho})^2$

The results, which are summarized in the above table, imply the following identities.

$\boldsymbol{c = \frac{2}{3}}$

$$F\left(\frac{1}{12},\frac{1}{12},\frac{2}{3};J(\tau)\right) = \begin{cases} \frac{\tau-\bar{\rho}}{i\sqrt{3}}\left(\frac{\eta(\tau)}{\eta(\rho)}\right)^2 & \text{if } \tau \in C_{J,\rho} \\ \frac{\tau+\rho}{i\sqrt{3}}\left(\frac{\eta(\tau)}{\eta(-\bar{\rho})}\right)^2 & \text{if } \tau \in C_{J,-\bar{\rho}} \end{cases}$$

$$F\left(\frac{11}{12},\frac{11}{12},\frac{4}{3};J(\tau)\right) = \begin{cases} \frac{\tau-\bar{\rho}}{i\sqrt{3}}(1-J(\tau))^{-\frac{1}{2}}\left(\frac{\eta(\tau)}{\eta(\rho)}\right)^2 & \text{if } \tau \in C_{J,\rho} \\ \frac{\tau+\rho}{i\sqrt{3}}(1-J(\tau))^{-\frac{1}{2}}\left(\frac{\eta(\tau)}{\eta(-\bar{\rho})}\right)^2 & \text{if } \tau \in C_{J,-\bar{\rho}} \end{cases} \tag{12}$$

$$F\left(\frac{1}{12},\frac{7}{12},\frac{2}{3};\frac{J(\tau)}{J(\tau)-1}\right) = \begin{cases} \frac{\tau-\bar{\rho}}{i\sqrt{3}}(1-J(\tau))^{\frac{1}{12}}\left(\frac{\eta(\tau)}{\eta(\rho)}\right)^2 & \text{if } \tau \in C_{\frac{J}{J-1},\rho} \\ \frac{\tau+\rho}{i\sqrt{3}}(1-J(\tau))^{\frac{1}{12}}\left(\frac{\eta(\tau)}{\eta(-\bar{\rho})}\right)^2 & \text{if } \tau \in C_{\frac{J}{J-1},-\bar{\rho}}. \end{cases}$$

Note 2. The identity for $F(\frac{1}{12},\frac{7}{12},\frac{2}{3};z)$ on $C_{\frac{J}{J-1},\rho}$ was already known by Beukers and Wolfart [6].

$\boldsymbol{c = \frac{4}{3}}$

$$F\left(\frac{5}{12}, \frac{5}{12}, \frac{4}{3}; J(\tau)\right) = \begin{cases} res_\rho(J^{-\frac{1}{3}})^{-1}(\tau-\rho)J(\tau)^{-\frac{1}{3}}(\frac{\eta(\tau)}{\eta(\rho)})^2 & \text{if } \tau \in C_{J,\rho} \\ res_{-\bar\rho}(J^{-\frac{1}{3}})^{-1}(\tau+\bar\rho)J(\tau)^{-\frac{1}{3}}(\frac{\eta(\tau)}{\eta(-\bar\rho)})^2 & \text{if } \tau \in C_{J,-\bar\rho} \end{cases}$$

$$\begin{aligned} &F\left(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}; J(\tau)\right) \\ &= \begin{cases} res_\rho(J^{-\frac{1}{3}})^{-1}(\tau-\rho)J(\tau)^{-\frac{1}{3}}(1-J(\tau))^{-\frac{1}{2}}(\frac{\eta(\tau)}{\eta(\rho)})^2 & \text{if } \tau \in C_{J,\rho} \\ res_{-\bar\rho}(J^{-\frac{1}{3}})^{-1}(\tau+\bar\rho)J(\tau)^{-\frac{1}{3}}(1-J(\tau))^{-\frac{1}{2}}(\frac{\eta(\tau)}{\eta(-\bar\rho)})^2 & \text{if } \tau \in C_{J,-\bar\rho} \end{cases} \\ &F\left(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; \frac{J(\tau)}{J(\tau)-1}\right) \\ &= \begin{cases} res_\rho(J^{-\frac{1}{3}})^{-1}(\tau-\rho)J(\tau)^{-\frac{1}{3}}(1-J(\tau))^{\frac{5}{12}}(\frac{\eta(\tau)}{\eta(\rho)})^2 & \text{if } \tau \in C_{\frac{J}{J-1},\rho} \\ res_{-\bar\rho}(J^{-\frac{1}{3}})^{-1}(\tau+\bar\rho)J(\tau)^{-\frac{1}{3}}(1-J(\tau))^{\frac{5}{12}}(\frac{\eta(\tau)}{\eta(-\bar\rho)})^2 & \text{if } \tau \in C_{\frac{J}{J-1},-\bar\rho}. \end{cases} \end{aligned} \tag{13}$$

3.3. Solutions at $J = \infty$

The invariance under $\tau \mapsto \tau + 1$ of J and η implies $A = 0$. The function $J^{-\frac{1}{6}-\alpha}(1-J)^{\frac{1}{4}-\beta}\eta^2$ has a pole of order $-\frac{1}{6} - \alpha + \frac{1}{4} - \beta - \frac{1}{12} = 0$ at infinity with constant term equal to $1728^{-\frac{1}{12}}$, because $\alpha + \beta = 0$ in each case. This implies $B = 1728^{\frac{1}{12}} = 12^{\frac{1}{4}}$. We summarize these informations in a table and give the consequent identities.

c	solutions	$\alpha+\beta$	τ_0	γ_{τ_0}	A	B
1	$\Omega_{\infty,a}(J) = \Omega_{\infty,b}(J)$	0	∞	$\tau \mapsto \tau+1$	0	$12^{\frac{1}{4}}$

$\boldsymbol{c = 1}$

$$\begin{aligned} \forall \tau \in C_{\frac{1}{J},\infty}: \quad & F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{1}{J(\tau)}\right) = \sqrt[4]{12}J(\tau)^{\frac{1}{12}}\eta(\tau)^2 \\ & F\left(\frac{7}{12}, \frac{11}{12}, 1; \frac{1}{J(\tau)}\right) = \sqrt[4]{12}J(\tau)^{\frac{7}{12}}(1-J(\tau))^{-\frac{1}{2}}\eta(\tau)^2 \end{aligned} \tag{14}$$

$$\begin{aligned} \forall \tau \in C_{\frac{1}{1-J},\infty}: \quad & F\left(\frac{1}{12}, \frac{7}{12}, 1; \frac{1}{1-J(\tau)}\right) = \sqrt[4]{12}(1-J(\tau))^{\frac{1}{12}}\eta(\tau)^2 \\ & F\left(\frac{5}{12}, \frac{11}{12}, 1; \frac{1}{1-J(\tau)}\right) = \sqrt[4]{12}J(\tau)^{-\frac{1}{3}}(1-J(\tau))^{\frac{5}{12}}\eta(\tau)^2. \end{aligned} \tag{15}$$

Remark 5. We curiously note that in the identities with $v(J) = \frac{1}{J}$ resp. $\frac{1}{1-J}$, the exponent of $J(\tau)$ resp. $1 - J(\tau)$ is equal to the smallest parameter of the series. (Thanks to R. Pink for this observation.)

Remark 6. Similar identities can be worked out when the hypergeometric differential equation is the Picard–Fuchs differential equation of some family of elliptic curves. We give an example for a hypergeometric series whose monodromy group is of type (∞, ∞, ∞), i.e. is isomorphic to the principal congruence subgroup $\Gamma(2)$ of level 2. For τ in the connected component of $\{\tau \in \mathbb{H}; |\lambda(\tau)| < 1\}$ which contains ∞, we have

$$F\left(\frac{1}{2}, \frac{1}{2}, 1; \lambda(\tau)\right) = 2^{\frac{1}{2}} 3^{-\frac{1}{4}} \pi g_2(\tau)^{-\frac{1}{4}} (\lambda^2(\tau) - \lambda(\tau) + 1)^{\frac{1}{4}}.$$

Remark 7. Relations between hypergeometric series and Eisenstein series are classicaly known (see for example [23]). Local relations follow also directly from identities (14) and (15). Indeed, we have

$$\forall \tau \in C_{\frac{1}{J}, \infty} : F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{1}{J(\tau)}\right)^4 = E_2(\tau) \quad \text{and}$$

$$\forall \tau \in C_{\frac{1}{1-J}, \infty} : F\left(\frac{1}{12}, \frac{7}{12}, 1; \frac{1}{1 - J(\tau)}\right)^6 = E_3(\tau),$$

where E_k is the normalized Eisenstein series of weight $2k$.

4. Consequences of the identities

This section gives some consequences of the identities proved in Section 3. Of most importance for the determination of the exceptional sets is Section 4.1, which determines, in each case with $c < 1$, an infinite subset of the exceptional set. In Section 4.2 some values of $J(mi)$ and $(\frac{\eta(mi)}{\eta(i)})^2$ and some algebraic evaluations of $F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z)$ at some $z \in \bar{\mathbb{Q}}$ are calculated. These evaluations provide examples for Kroneckers Jugendtraum. Section 4.3 gives some relations between values of transcendental functions (Proposition 4.5) and also some transcendental evaluations of $F(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; z)$ at algebraic points.

4.1. *Infinite subsets of nontrivial exceptional sets*

For the determination of infinite subsets of the exceptional sets, we will use the two following classical facts on modular functions.

(F1) Let $A, B, D \in \mathbb{Z}$ with $AD > 0$ and for $\tau \in \mathbb{H}$, set $\varphi(\tau) = (\frac{\eta(\frac{A\tau+B}{D})}{\eta(\tau)})^2$, then the function φ is integral over $\mathbb{Z}[1728J]$ (see [18, 12.2, Theorem 2]).

(F2) If $\tau \in \mathbb{H}$ is imaginary quadratic, then $J(\tau) \in \bar{\mathbb{Q}}$ (see [18, 5.2, Theorem 4]).

$\boldsymbol{c = \frac{1}{2}}$

Let $v \in \{1 - J, \frac{J-1}{J}\}$. If $\tau \in \mathbb{Q}(i) \cap \{w \in \mathbb{H}; |v(J)(w)| < 1\}$, then by (F2), we have $v(J)(\tau) \in \bar{\mathbb{Q}}$. Moreover, τ can be written as $\frac{Ai+B}{D}$ with $AD > 0$ because $\tau \in \mathbb{Q}(i) \cap \mathbb{H}$. By (F2), this implies $(\frac{\eta(\tau)}{\eta(i)})^2 \in \bar{\mathbb{Q}}$. By the identities (10), we have

Lemma 4.1. *For* $\tau \in \mathbb{Q}(i) \cap \{w \in \mathbb{H}; |1 - J(w)| < 1\}$, *we have* $1 - J(\tau) \in \bar{\mathbb{Q}}$ *and*

$$F\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}; 1 - J(\tau)\right) \in \bar{\mathbb{Q}} \quad \textit{and} \quad F\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{2}; 1 - J(\tau)\right) \in \bar{\mathbb{Q}}.$$

Lemma 4.2. *For* $\tau \in \mathbb{Q}(i) \cap \{w \in \mathbb{H}; |\frac{J(w)-1}{J(w)}| < 1\}$, *we have* $\frac{J(\tau)-1}{J(\tau)} \in \bar{\mathbb{Q}}$ *and*

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{J(\tau) - 1}{J(\tau)}\right) \in \bar{\mathbb{Q}}.$$

Proof. By the above explanations, it is clear that these results hold for each τ in $\mathbb{Q}(i) \cap C_{v(J),i}$. Further, for any τ_0 congruent to $i \bmod SL_2(\mathbb{Z})$, there exists an identity like (9) valid in the neighbourhood of τ_0. This identity will defer from that of (10) only up to an algebraic factor which comes out by the transformation properties of the η-function. We conclude by applying the same reasoning to that identity. □

$\boldsymbol{c = \frac{2}{3}}$

It goes very similarly as above. If $\tau \in \mathbb{Q}(i\sqrt{3}) \cap \{w \in \mathbb{H}; |v(J)(w)| < 1\}$, then $v(J)(\tau) \in \bar{\mathbb{Q}}$ and $(\frac{\eta(\tau)}{\eta(\rho)})^2 \in \bar{\mathbb{Q}}$ by (F1) and (F2). The algebraicity at the points of $\tau \in \mathbb{Q}(i\sqrt{3}) \cap (C_{v(J),\rho} \cup C_{v(J),-\bar{\rho}})$ follows then directly from the identities (12). The algebraicity of the values at the other points is obtained as in the above proof.

Lemma 4.3. *For* $\tau \in \mathbb{Q}(i\sqrt{3}) \cap \{w \in \mathbb{H}; |J(w)| < 1\}$, *we have* $J(\tau) \in \bar{\mathbb{Q}}$ *and*

$$F\left(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}; J(\tau)\right) \in \bar{\mathbb{Q}} \quad \text{and} \quad F\left(\frac{7}{12}, \frac{7}{12}, \frac{2}{3}; J(\tau)\right) \in \bar{\mathbb{Q}}.$$

Lemma 4.4. *For* $\tau \in \mathbb{Q}(i\sqrt{3}) \cap \{w \in \mathbb{H}; |\frac{J(w)}{J(w)-1}| < 1\}$, *we have* $\frac{J(\tau)}{J(\tau)-1} \in \bar{\mathbb{Q}}$ *and*

$$F\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}; \frac{J(\tau)}{J(\tau) - 1}\right) \in \bar{\mathbb{Q}}.$$

4.2. *Some algebraic evaluations of* $F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z)$

The following chart gives the values of $J(mi)$ and $(\frac{\eta(mi)}{\eta(i)})^2$ for some m. These values were computed in [4, 2.6.2] with the help of *Mathematica* in using some classical results on modular equations (to be found in [25]).

m	$J(mi)$
2	$\frac{(8^3+16)^3}{2^6 3^3 8^3} = \frac{(11)^3}{2^3}$
3	$\frac{((1+\sqrt{3})^8-4)^3}{2^2 3^3 (1+\sqrt{3})^8} = \frac{133283}{3} + \frac{2^3 7^2 19 \cdot 31\sqrt{3}}{3^2}$
4	$\frac{(2^{\frac{13}{2}}(1+\sqrt{2})^6+1)^3}{2^{\frac{9}{2}} 3^3 (1+\sqrt{2})^6} = \frac{2^2 5 \cdot 181 \cdot 210319 + 3^4 7^2 (11)^2 19 \cdot 59\sqrt{2}}{2^5}$
5	$\frac{((1+\sqrt{5})^{24}-2^{22})^3}{2^{42} 3^3 (1+\sqrt{5})^{24}} = 1637 \cdot 2659 \cdot 2927 + 2^4 3^4 7^2 \cdot 23 \cdot 47 \cdot 83\sqrt{5}$
6	$\frac{((2+\sqrt{3}+\sqrt{9+6\sqrt{3}})^8+2)^3}{3^3(2+\sqrt{3}+\sqrt{9+6\sqrt{3}})^8}$
7	$\frac{((2+\sqrt{7}+\sqrt{7+4\sqrt{7}})^{12}-2^{10})^3}{2^{18} 3^3 (2+\sqrt{7}+\sqrt{7+4\sqrt{7}})^{12}}$
8	$\frac{(2^7\Upsilon+1)^3}{2^5 3^3 \Upsilon}$, where $\Upsilon := 2^4 \cdot 257 \cdot 2441 + 2^2 3^2 \cdot 11 \cdot 17923\sqrt{2}$ $+3^2\sqrt{2^2 \cdot 3 \cdot 221047 \cdot 937823 + 11 \cdot 17 \cdot 139 \cdot 67672987\sqrt{2}}$
10	$\frac{((1+\sqrt{5}+\sqrt{10+6\sqrt{5}})^{24}+2^{31})^3}{2^{60} 3^3 (1+\sqrt{5}+\sqrt{10+6\sqrt{5}})^{24}}$

m	$\left(\frac{\eta(mi)}{\eta(i)}\right)^2$
2	$2^{-\frac{3}{4}}$
3	$3^{-\frac{5}{6}}(-3+2\sqrt{3})^{\frac{1}{6}}$
4	$2^{-\frac{13}{8}}(1+\sqrt{2})^{-\frac{1}{2}}$
5	$\frac{-1+\sqrt{5}}{2\cdot 5}$
6	$\frac{(-3+2\sqrt{3})^{\frac{1}{6}}}{2^{\frac{1}{4}}3^{\frac{5}{6}}(2+\sqrt{3}+\sqrt{9+6\sqrt{3}})^{\frac{2}{3}}}$
7	$\frac{1}{7}\sqrt{-\frac{7}{2}+\sqrt{7}+\frac{1}{2}\sqrt{-7+4\sqrt{7}}}$
8	$2^{-\frac{61}{24}}(1+\sqrt{2})^{-\frac{1}{2}}\Big(2^4\cdot 257\cdot 2441+2^2 3^2\cdot 11\cdot 17923\sqrt{2}$ $+3^2\sqrt{2^2\cdot 3\cdot 221047\cdot 937823+11\cdot 17\cdot 139\cdot 67672987\sqrt{2}}\Big)^{-\frac{1}{12}}$
10	$\frac{2^{\frac{5}{4}}(-1+\sqrt{5})}{5(1+\sqrt{5}+\sqrt{10+6\sqrt{5}})^2}$

The following algebraic evaluations of $F(\frac{1}{12},\frac{5}{12},\frac{1}{2};z)$ were calculated by inserting the values given in the previous charts into the identity for this series given in (10).

For $\tau = 2i$ we get,

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{3^3 7^2}{11^3}\right) = \frac{3\cdot 11^{\frac{1}{4}}}{2^2}$$

$\tau = 3i$,

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{2^3 7^2(2^2\cdot 11093 - 3\cdot 19\cdot 31\sqrt{3})}{(11\cdot 23)^3}\right) = \frac{2}{3}(3\cdot 7+2^2\cdot 5\sqrt{3})^{\frac{1}{4}}$$

$\tau = 4i$,

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{3^3 7^2 11^2(83967 - 2^4\cdot 3\cdot 19\cdot 59\sqrt{2})}{(23\cdot 47)^3}\right) = \frac{5}{2^3}(7\cdot 13+2^2\cdot 3\cdot 5\sqrt{2})^{\frac{1}{4}}$$

$\tau = 5i$,

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{2^4 3^3 7^2 (2^3\cdot 5\cdot 5237\cdot 22067 - 3\cdot 23\cdot 47\cdot 83\sqrt{5})}{(11\cdot 59\cdot 71)^3}\right) = \frac{3}{5}(7\cdot 23 + 2^3\cdot 3\cdot 5\sqrt{5})^{\frac{1}{4}}$$

$\tau = 6i$,

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{(2+(2+\sqrt{3}+\sqrt{9+6\sqrt{3}})^8)^3 - 27(2+\sqrt{3}+\sqrt{9+6\sqrt{3}})^8}{(2+(2+\sqrt{3}+\sqrt{9+6\sqrt{3}})^8)^3}\right) = \frac{7}{2^2\cdot 3}\left(3\cdot 7\cdot 11 + 2^2\cdot 5\cdot 7\sqrt{3} + 2^3\cdot 3\cdot 5\sqrt{2\cdot 3 + \frac{7\sqrt{3}}{2}}\right)^{\frac{1}{4}}$$

$\tau = 7i$,

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{((2+\sqrt{7}+\sqrt{7+4\sqrt{7}})^{12} - 2^{10})^3 - 2^{18}3^3(2+\sqrt{7}+\sqrt{7+4\sqrt{7}})^{12}}{((2+\sqrt{7}+\sqrt{7+4\sqrt{7}})^{12} - 2^{10})^3}\right)$$
$$\frac{2^2}{7}\left(7\cdot 43 + 2^3\cdot 3\cdot 5\sqrt{7} + 2^2\cdot 3\cdot 5\cdot 7\sqrt{3+\frac{8}{\sqrt{7}}}\right)^{\frac{1}{4}}$$

$\tau = 8i$,

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{(2^7\Upsilon+1)^3 - 2^5 3^3\Upsilon}{(2^7\Upsilon+1)^3}\right) = \frac{3^{\frac{7}{4}}(2^7\Upsilon+1)^{\frac{1}{4}}}{2^{\frac{95}{24}}(1+\sqrt{2})^{\frac{1}{2}}\Upsilon^{\frac{1}{6}}}, \quad \text{where}$$
$$\Upsilon = 2^4\cdot 257\cdot 2441 + 2^2 3^2\cdot 11\cdot 17923\sqrt{2} + 3^2\sqrt{2^2\cdot 3\cdot 221047\cdot 937823 + 11\cdot 17\cdot 139\cdot 67672987\sqrt{2}}$$

and for $\tau = 10i$,

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{(\Sigma^{24}+2^{31})^3 - 2^{60}3^3\Sigma^{24}}{(\Sigma^{24}+2^{31})^3}\right) = \frac{11(-1+\sqrt{5})(\Sigma^{24}+2^{31})^{\frac{1}{4}}}{2^{\frac{21}{4}}\cdot 3^{\frac{1}{4}}\cdot 5\Sigma^4}, \quad \text{where}$$
$$\Sigma := 1+\sqrt{5}+\sqrt{10+6\sqrt{5}}.$$

Remark 8. The evaluation at $\tau = 2i$ and $3i$ were given by Beukers and Wolfart [6] and Flach [13], respectively. Joyce and Zucker [16] produced the evaluations at $\tau = 4i$ and $5i$ by a different method. The evaluations at $\tau = 6i,\ 7i,\ 8i,\ 10i$ are new.

4.3. Relations between values of transcendental functions and evaluations of $F(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; z)$

The comparison between the work [16] of Joyce and Zucker and some of our results provides some amusing relations.

Proposition 4.5. *Let J be the modular invariant, Γ be the classical Gamma function and η be the Dedekind Eta-function. Then we have*

$$res_i((1-J)^{-\frac{1}{2}}) = \frac{2^3 i\pi^2}{\sqrt{3}\Gamma(\frac{1}{4})^4} \tag{16}$$

and

$$\eta(i)^2 = \frac{\Gamma(\frac{1}{4})^2}{2^2\pi^{\frac{3}{2}}}. \tag{17}$$

Proof. Joyce and Zucker [15] proved the following relations

$$F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; \frac{1323}{1331}\right) = \frac{(11)^{\frac{7}{4}}}{336\pi^2}\Gamma\left(\frac{1}{4}\right)^4 \tag{18}$$

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{J(\tau)-1}{J(\tau)}\right) = \frac{2\pi^{\frac{3}{2}}}{i\Gamma(\frac{1}{4})^2}(\tau+i)J(\tau)^{\frac{1}{12}}\eta(\tau)^2. \tag{19}$$

We use $\frac{1323}{1331} = \frac{J(2i)-1}{J(2i)}$ to compare (18) to our identity (11) for $F(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; \frac{J(\tau)-1}{J(\tau)})$ evaluated at $\tau = 2i$. This yields relation (16). The comparison of (19) to the identity for $F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z)$ given in (10) gives relation (17). $\square$

Note 3. We thank the referee for noting that relation (16) could be obtained in a standard way along modular forms. Indeed, a direct calculation shows that the residue of $(1-J)^{-\frac{1}{2}}$ at i is the quotient of a period by a quasi-period on E_J with $J = 1$. In particular, this is a transcendental number by Wüstholz's result (W2) quoted in Section 5. Further, this corresponds via (16) to the algebraic independence of π and $\Gamma(\frac{1}{4})$ proved by Chudnovsky.

Using relation (16) together with the values of $J(mi)$ and $(\frac{\eta(mi)}{\eta(i)})^2$ calculated in Section 4.2, we compute as examples the evaluations of $F(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; z)$ at $\tau = 3i, 4i, 5i$. We obtain, for $\tau = 3i$,

$$F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2};\frac{2^3 7^2(2^2\cdot 11093-3\cdot 19\cdot 31\sqrt{3})}{(11\cdot 23)^3}\right)$$
$$=\left(\frac{\Gamma(\frac{1}{4})^4}{\pi^2}\right)\frac{(3\cdot 47\cdot 361359611+2^2 7^3\cdot 401\cdot 77899\sqrt{3})^{\frac{1}{4}}}{2^3\cdot 7\cdot 11\sqrt{6}}$$

$\tau = 4i$,

$$F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2};\frac{3^3 7^2 11^2(83967-2^4\cdot 3\cdot 19\cdot 59\sqrt{2})}{(23\cdot 47)^3}\right)$$
$$=\left(\frac{\Gamma(\frac{1}{4})^4}{\pi^2}\right)\frac{(733\cdot 3074274799+2^2\cdot 3\cdot 11\cdot 99746711\sqrt{2})^{\frac{1}{4}}}{2^5\cdot 7\cdot 11}$$

$\tau = 5i$,

$$F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2};\frac{2^4 3^3 7^2(2^3\cdot 5\cdot 5237\cdot 22067-3\cdot 23\cdot 47\cdot 83\sqrt{5})}{(11\cdot 59\cdot 71)^3}\right)=\left(\frac{\Gamma(\frac{1}{4})^4}{\pi^2}\right)$$
$$\cdot\frac{(23\cdot 43\cdot 147600623\cdot 4715541403+2^3 3^2 7^3\cdot 190829\cdot 108837819119\sqrt{5})^{\frac{1}{4}}}{2^3\cdot 3\cdot 5^{\frac{5}{4}}\cdot 7\cdot 11\cdot 19}.$$

5. Determination of the exceptional sets

For the triples (a,b,c) considered in Section 3 for which $c\neq 1$, we give a necessary condition for $z\in\mathbb{C}$ to lie in $E(a,b,c)$. Here will be used the abelian varieties constructed in Section 2 and the mapping properties of the quotient of two linearly independent solutions to $Equ(a,b,c)$ given in Section 1. The key tool to get the condition on z is the following consequence of Wüstholz's Analytic Subgroup Theorem [30, Hauptsatz].

Proposition 5.1 (Wüstholz [29, Satz 2]). *Let $A_1,\dots,A_m$ be pairwise nonisogenous simple abelian varieties defined over $\bar{\mathbb{Q}}$. For each $j=1,\dots,m$, set $n_j=\dim A_j$ and choose $0\neq\gamma_j\in H_1(A_j,\mathbb{Z})$. Let $\omega_1^{(j)},\dots,\omega_{n_j}^{(j)}$ and $\eta_1^{(j)},\dots,\eta_{n_j}^{(j)}$ be bases of differential forms on A_j of first and second kind, respectively. Then the vector space generated over $\bar{\mathbb{Q}}$ by the numbers*

$$\int_{\gamma_j}\omega_k^{(j)},\quad \int_{\gamma_j}\eta_k^{(j)},\quad 1\geqslant j\geqslant m,\ \ 1\geqslant k\geqslant n_j,$$

has dimension $2\sum_{j=1}^m n_j$.

We will use the two following consequences of this result.

(W1) Let A_1, A_2 be two abelian varieties defined over $\bar{\mathbb{Q}}$. For $i = 1, 2$, let $\int_{\gamma_i} \omega_i$ be a period of the first kind on A_i. If $\frac{\int_{\gamma_1} \omega_1}{\int_{\gamma_2} \omega_2} \in \bar{\mathbb{Q}}$, then there exist simple abelian subvarieties A_i' of A_i, $i = 1, 2$, such that A_1' is isogenous to A_2'.

(W2) Let A_1, A_2 be two abelian varieties defined over $\bar{\mathbb{Q}}$. Let $\int_{\gamma_1} \omega_1$ be a period of first kind on A_1 and $\int_{\gamma_2} \omega_2$ be a period of second kind on A_2, then

$$\frac{\int_{\gamma_1} \omega_1}{\int_{\gamma_2} \omega_2} \notin \bar{\mathbb{Q}}.$$

The following table gives the denominators of the integral representation of the considered hypergeometric series in dependence of the value of c.

c	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{3}{2}$
den.	$B(\frac{1}{12}, \frac{5}{12})$	$B(\frac{1}{12}, \frac{7}{12})$	$B(\frac{1}{12}, \frac{11}{12})$ or $B(\frac{5}{12}, \frac{7}{12})$	$B(\frac{5}{12}, \frac{11}{12})$	$B(\frac{7}{12}, \frac{11}{12})$

(20)

Conditions (5) of Section 2 for the regularity of differential forms show that the numerators of the hypergeometric series with $c = \frac{4}{3}$ and $\frac{3}{2}$ are of the first kind when $z \neq 0$. One can verify that the denominators are of second kind (see the holomorphy conditions in [4, 1.6.3]). Hence, (W2) implies the transcendence of the quotients for which $z \neq 0$ and we have

$$E\left(\frac{7}{12}, \frac{7}{12}, \frac{3}{2}\right) = E\left(\frac{11}{12}, \frac{11}{12}, \frac{3}{2}\right) = E\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}\right) = \{0\} \tag{21}$$

$$E\left(\frac{5}{12}, \frac{5}{12}, \frac{4}{3}\right) = E\left(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}\right) = E\left(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}\right) = \{0\}. \tag{22}$$

Note 4. These are in fact special issues of a more general result. Indeed, if $a, b, c \in \mathbb{Q}$ satisfy $a + 1 > c > b > 0$, $a < 1$ and $c > 1$, then $E(a, b, c) = \{0\}$. The conditions on a, b, c ensure the period in the numerator to be of the first kind, whereas the denominator be of second kind. Hence, the result follows from (W2). We wish to thank the referee for pointing this out.

Remark 9. In the cases with $c = 1$, the denominator has a nonzero residue, hence is of the third kind. We do not know what the nature of the exceptional set is in that case.

For each triple (a, b, c) with $c < 1$, we have $N = 12$ and the dimension formula (6) shows that all the V_n with $(n, N) = 1$ are 1-dimensional. Let V_n be generated by ω_n.

The construction in Section 2 associates to each (a,b,c) and each $z \in \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$ an abelian variety $T_{abc}(z)$ of dimension $\varphi(12) = 4$ which is isogenous to a quotient $\mathbb{C}^4/\Lambda_{abc}(z)$, where

$$\Lambda_{abc}(z) := \left\{ \left(\sigma_n(u) \int_{\gamma_{01}} \omega_n + \sigma_n(v) \int_{\gamma_{\frac{1}{z}\infty}} \omega_n \right)_{n=1,5,7,11} ; u, v \in \mathbb{Z}[\zeta_{12}] \right\}.$$

Since $\int_{\gamma_{01}} \omega_n$ and $\int_{\gamma_{\frac{1}{z}\infty}} \omega_n$ are linearly independent solutions to $Equ(a,b,c)$ (see [15, 2.4.5]), their quotient $D_n(z) := \int_{\gamma_{\frac{1}{z}\infty}} \omega_n / \int_{\gamma_{01}} \omega_n$ is a Schwarz triangle function. One verifies that $D_1 = D_{abc}$. Hence, $D := D_1$ maps the upper half plane into the interior of a hyperbolic triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}, \infty$. The vertices can be calculated thanks to the results of Section 1. The triangle depends only on the transformation $v(J)$ for which $J^{\alpha}(1-J)^{\beta} F(a,b,c;v(J))$ is solution in (7). The following chart gives the vertices and the corresponding angles of the triangle in dependence of $v(J)$.

c	$\nu(J)$	z	$D(z)$	angle
$\frac{1}{2}$	$1-J$	0	0	$\frac{\pi}{2}$
		1	$\bar{\rho}\tan\frac{\pi}{3}$	$\frac{\pi}{3}$
		∞	1	0
	$\frac{J-1}{J}$	0	0	$\frac{\pi}{2}$
		1	-1	0
		∞	$-i\tan\frac{\pi}{12}$	$\frac{\pi}{3}$
$\frac{2}{3}$	J	0	0	$\frac{\pi}{3}$
		1	$-i\tan\frac{\pi}{12}$	$\frac{\pi}{2}$
		∞	1	0
	$\frac{J}{J-1}$	0	0	$\frac{\pi}{3}$
		1	-1	0
		∞	$\bar{\rho}\tan\frac{\pi}{3}$	$\frac{\pi}{2}$

(23)

Each transformation $v(J)$ maps either the left-hand side $\mathfrak{L} := \{\tau \in \mathbb{H}; |\tau| > 1, -\frac{1}{2} < Re(\tau) < 0\}$ or the right-hand side $\mathfrak{R} := \{\tau \in \mathbb{H}; |\tau| > 1, 0 < Re(\tau) < \frac{1}{2}\}$ of the standard fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ by fractional linear transformations. Moreover, the composition $D \circ v(J)$ is a conformal mapping of $\mathfrak{L}$ resp. $\mathfrak{R}$ to the interior $D(\mathbb{H})$ of the hyperbolic triangle. By the uniqueness part of the Riemann mapping theorem, there exists a unique inverse conformal mapping μ to $D \circ v(J)$.

The following table gives the map μ for each $\nu(J)$ and some relations between the Schwarz triangle functions D_n.

c		$\nu(J)$	μ	relations
$\frac{1}{2}$	$\mathfrak{L}$	$1-J$	$u \mapsto \frac{(1-i)u-i}{u-1}$	$D_1 = D_5 = D_7^{-1} = D_{11}^{-1}$
		$\frac{J-1}{J}$	$u \mapsto -i\frac{u-1}{u+1}$	
$\frac{2}{3}$	$\mathfrak{R}$	J	$u \mapsto \frac{1-i\sqrt{3}}{2}\frac{u+1}{u-1}$	$D_1 = D_5^{-1} = D_7 = D_{11}^{-1}$
		$\frac{J}{J-1}$	$u \mapsto \frac{-1-i\sqrt{3}}{2}\frac{u-1}{u+1}$	

Remark 10. Using the multiplication theorem [26, 12·15] for the Gamma function

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdot\ldots\cdot\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}-nz}\Gamma(nz),$$

one can show

$$B\left(\frac{1}{12},\frac{5}{12}\right) = 2\cdot 3^{\frac{1}{4}}B\left(\frac{1}{2},\frac{1}{4}\right),$$

$$B\left(\frac{1}{12},\frac{7}{12}\right) = 2^{\frac{5}{6}}B\left(\frac{1}{2},\frac{1}{6}\right).$$

By table (20), the above numbers are the denominators of the hypergeometric series with $c<1$. Since we care about these denominators only up to algebraic factors, we can consider the period $B(\frac{1}{2},\frac{1}{4})$ when $c=\frac{1}{2}$ and $B(\frac{1}{2},\frac{1}{6})$ when $c=\frac{2}{3}$.

$\boldsymbol{c=\frac{1}{2}}$

By Remark 4, Section 2, $B(\frac{1}{2},\frac{1}{4})$ is a period on an abelian variety $T_{\mathfrak{L},0}$ of dimension 1 (hence simple) whose endormorphism ring contains $\mathbb{Z}[\zeta_4]$. Thus, $T_{\mathfrak{L},0}$ is isogenous to the torus $\mathbb{C}/\Lambda_{\mathfrak{L},0}$, where $\Lambda_{\mathfrak{L},0} = \mathbb{Z}\oplus\tau\mathbb{Z}$ with $\tau\in\mathbb{Q}(i)$. On the other hand, $T_{abc}(z) =:$ $T_{\mathfrak{L},z}$ is isogenous to the quotient of $\mathbb{C}^4$ by the lattice

$$\Lambda_{\mathfrak{L},z} = \{(u+vD(z), \sigma_5(u)+\sigma_5(v)D(z),\\ \sigma_7(u)D(z)+\sigma_7(v), \sigma_{11}(u)D(z)+\sigma_{11}(v)); u,v\in\mathbb{Z}[\zeta_{12}]\}.$$

Suppose now that $z\in\bar{\mathbb{Q}}$ and $F(a,b,\frac{1}{2};z)\in\bar{\mathbb{Q}}$, then $T_{\mathfrak{L},0}$ and $T_{\mathfrak{L},z}$ are defined over $\bar{\mathbb{Q}}$ and (W1) implies the existence of an isogeny of $T_{\mathfrak{L}}(0)$ to a simple factor of $T_{\mathfrak{L}}(z)$. In particular, there exits a $\bar{\mathbb{Q}}$-linear homomorphism $\Phi:\mathbb{C}\to\mathbb{C}^4$ such that $\Phi(\Lambda_{\mathfrak{L},0})\subset\Lambda_{\mathfrak{L},z}$. Writing $\Phi(1)$ and $\Phi(\tau)$ as elements of $\Lambda_{\mathfrak{L},z}$, the condition $\Phi(\tau)=\tau\Phi(1)$ implies

$D(z) \in \mathbb{Q}(i)$. Since μ is a rational function with coefficients in $\mathbb{Q}(i)$, we have $(\mu \circ D)(z) \in \mathbb{Q}(i)$. Using $z = (v(J) \circ \mu \circ D)(z)$, we get

Lemma 5.2. *For* $|z| < 1$, *we have*

$$
\begin{aligned}
&z \in \bar{\mathbb{Q}}, \quad F\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}; z\right) \in \bar{\mathbb{Q}} \Rightarrow \exists \tau \in \mathbb{Q}(i) \cap \mathbb{H} \quad s.t.\ z = (1 - J)(\tau) \\
&z \in \bar{\mathbb{Q}}, \quad F\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{2}; z\right) \in \bar{\mathbb{Q}} \Rightarrow \exists \tau \in \mathbb{Q}(i) \cap \mathbb{H} \quad s.t.\ z = (1 - J)(\tau) \\
&z \in \bar{\mathbb{Q}}, \quad F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z\right) \in \bar{\mathbb{Q}} \Rightarrow \exists \tau \in \mathbb{Q}(i) \cap \mathbb{H} \quad s.t.\ z = \left(\frac{J-1}{J}\right)(\tau).
\end{aligned}
$$

$\boldsymbol{c = \frac{2}{3}}$

It goes very similarly as for the previous case. $B(\frac{1}{2}, \frac{1}{6})$ is a period on an abelian variety $T_{\mathfrak{R},0}$ of dimension 1 whose endomorphism ring contains $\mathbb{Z}[\zeta_6]$. Hence, $T_{\mathfrak{R},0}$ is isogenous to the torus $\mathbb{C}/\Lambda_{\mathfrak{R},0}$, where $\Lambda_{\mathfrak{R},0} = \mathbb{Z} \oplus \tau\mathbb{Z}$ with $\tau \in \mathbb{Q}(i\sqrt{3})$. $T_{\mathfrak{R},z}$ is isogenous to the quotient of $\mathbb{C}^4$ by the lattice

$$
\begin{aligned}
\Lambda_{\mathfrak{R},z} = \{&(u + vD(z), \sigma_5(u)D(z) + \sigma_5(v), \\
&\sigma_7(u) + \sigma_7(v)D(z), \sigma_{11}(u)D(z) + \sigma_{11}(v)); u, v \in \mathbb{Z}[\zeta_{12}]\}.
\end{aligned}
$$

Now, if $z \in \bar{\mathbb{Q}}$ and $F(a, b, \frac{2}{3}; z) \in \bar{\mathbb{Q}}$, then $T_{\mathfrak{R},0}$ and $T_{\mathfrak{R},z}$ are defined over $\bar{\mathbb{Q}}$ and (W1) implies the existence of an isogeny of $T_{\mathfrak{R}}(0)$ to a simple factor of $T_{\mathfrak{R}}(z)$. As above, this implies a condition on $D(z)$, which is here $D(z) \in \mathbb{Q}(i\sqrt{3})$. By the previous chart, the maps μ are here rational functions with coefficients in $\mathbb{Q}(i\sqrt{3})$, hence we have $(\mu \circ D)(z) \in \mathbb{Q}(i\sqrt{3})$. The function $v(J) \circ \mu \circ D$ being the identity on $\mathbb{H}$, we have shown

Lemma 5.3. *For* $|z| < 1$, *we have*

$$
\begin{aligned}
&z \in \bar{\mathbb{Q}}, \quad F\left(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}; z\right) \in \bar{\mathbb{Q}} \Rightarrow \exists \tau \in \mathbb{Q}(i\sqrt{3}) \cap \mathbb{H} \quad s.t.\ z = J(\tau) \\
&z \in \bar{\mathbb{Q}}, \quad F\left(\frac{7}{12}, \frac{7}{12}, \frac{2}{3}; z\right) \in \bar{\mathbb{Q}} \Rightarrow \exists \tau \in \mathbb{Q}(i\sqrt{3}) \cap \mathbb{H} \quad s.t.\ z = J(\tau) \\
&z \in \bar{\mathbb{Q}}, \quad F\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}; z\right) \in \bar{\mathbb{Q}} \Rightarrow \exists \tau \in \mathbb{Q}(i\sqrt{3}) \cap \mathbb{H} \quad s.t.\ z = \left(\frac{J}{J-1}\right)(\tau).
\end{aligned}
$$

Bringing together Lemmata 4.1, 4.2, 4.3, 4.4, 5.2, 5.3 and considering (21), (22), we can state

Theorem 5.4. *The exceptional sets* $E(a,b,c)=\{x\in\bar{\mathbb{Q}}; F(a,b,c;z)\in\bar{\mathbb{Q}}\}$ *of the hypergeometric series* $F(a,b,c;z)$ *with* $c\neq 1$ *appearing in Kummer's solutions* (8) *to the Picard–Fuchs differential equation*

$$\frac{d^2\Omega}{dJ^2}+\frac{1}{J}\frac{d\Omega}{dJ}+\frac{31J-4}{144J^2(J-1)^2}\Omega=0$$

of the universal family of elliptic curves (*cf. Section* 3) *are given as follows*:

$$E\left(\frac{7}{12},\frac{7}{12},\frac{3}{2}\right)=E\left(\frac{11}{12},\frac{11}{12},\frac{3}{2}\right)=E\left(\frac{7}{12},\frac{11}{12},\frac{3}{2}\right)=\{0\}$$

$$E\left(\frac{5}{12},\frac{5}{12},\frac{4}{3}\right)=E\left(\frac{11}{12},\frac{11}{12},\frac{4}{3}\right)=E\left(\frac{5}{12},\frac{11}{12},\frac{4}{3}\right)=\{0\}$$

$$\left.\begin{array}{l}E(\frac{1}{12},\frac{1}{12},\frac{1}{2}),\\ E(\frac{5}{12},\frac{5}{12},\frac{1}{2})\end{array}\right\}=\{z\in\bar{\mathbb{Q}}\cap D(0,1);\exists\tau\in\mathbb{Q}(i)\cap\mathbb{H}\ s.t.\ z=1-J(\tau)\}$$

$$E\left(\tfrac{1}{12},\tfrac{5}{12},\tfrac{1}{2}\right)=\left\{z\in\bar{\mathbb{Q}}\cap D(0,1);\exists\tau\in\mathbb{Q}(i)\cap\mathbb{H}\ s.t.\ z=\frac{J(\tau)-1}{J(\tau)}\right\}$$

$$\left.\begin{array}{l}E(\frac{1}{12},\frac{1}{12},\frac{2}{3})\\ E(\frac{7}{12},\frac{7}{12},\frac{2}{3})\end{array}\right\}=\{z\in\bar{\mathbb{Q}}\cap D(0,1);\exists\tau\in\mathbb{Q}(i\sqrt{3})\cap\mathbb{H}\ s.t.\ z=J(\tau)\}$$

$$E\left(\tfrac{1}{12},\tfrac{7}{12},\tfrac{2}{3}\right)=\left\{z\in\bar{\mathbb{Q}}\cap D(0,1);\exists\tau\in\mathbb{Q}(i\sqrt{3})\cap\mathbb{H}\ s.t.\ z=\frac{J(\tau)}{J(\tau)-1}\right\},$$

where $D(0,1)$ *denotes the open unit disc in* $\mathbb{C}$.

Note 5. The determinations of $E(\frac{1}{12},\frac{5}{12},\frac{1}{2})$ and $E(\frac{1}{12},\frac{7}{12},\frac{2}{3})$ were known by Beukers and Wolfart [6].

Remark 11. The nontrivial exceptional sets appearing in Theorem 5.4 are dense in $\bar{\mathbb{Q}}\cap D(0,1)$, because J induces a continuos bijection $\mathbb{H}/SL_2(\mathbb{Z})\to\mathbb{C}$. In particular, they are infinite.

Remark 12. It follows from the triviality of the exceptional sets in regard to identities (11) resp. (13) and to the facts (F1) and (F2) of Section 4 that the residue of $(1-J)^{-\frac{1}{2}}$ resp. $J^{-\frac{1}{3}}$ at any point congruent to i resp. $e^{\frac{2\pi i}{3}}$ modulo $SL_2(\mathbb{Z})$ is a transcendental number. See also Note 3 in Section 4.3.

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